On the question of the preferred mode in cellular thermal convection

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This paper modifies and refines earlier work of Palm (1960) concerning the finite-amplitude steady state of cellular convective motion attained when a horizontal layer of fluid becomes unstable as a result of being heated from below. The two non-linear ordinary differential equations to which the problem was reduced by Palm (under certain conditions) are given in a corrected form, and are then analysed in some detail. The principal conclusions are that, for the model considered, hexagonal convection cells *may* be the stable equilibrium state only if the variation of kinematic viscosity with temperature is sufficiently great. Under the same circumstances a two-dimensional roll cell is also possible, the initial conditions determining which state actually occurs. Although further work is indicated, it seems probable that in an actual experiment with sufficiently large kinematic-viscosity variation, the hexagonal cells are more likely to appear. The analysis enables conclusions to be drawn concerning the flow direction at the cell centre, and also shows that a disturbance of sufficient magnitude may grow even though the situation is a stable one by linearized theory. Comparison with experiment is discussed.

1. Introduction

In a recent paper, Palm (1960) has studied the non-linear interaction of two special disturbances in thermal convection, in order to discover what combination of them might be 'preferred' at finite amplitudes. Palm also considered the direction of flow in the cells, a property which was determined at the same time as the question of the preferred mode. 'Free' boundary conditions both above and below were assumed and the kinematic viscosity was allowed to vary with temperature; the work showed the latter dependence to be of great importance.

As his two original disturbances, chosen from those allowed by linearized theory, Palm used the particular cells which, in a certain linear combination, produce a hexagonal cell. (Other reasons for choosing the original disturbances are considered below.) From his analysis he suggested that the hexagonal cell is the preferred mode at finite amplitudes, in the sense that the two disturbances considered interact as they grow in amplitude in such a way as to yield the hexagonal cell as the ultimate result. There are two criticisms which may be levelled against Palm's conclusions: (i) his equations should include as special cases, and agree with, the results of earlier calculations by Malkus & Veronis (1958) and Gorkov (1957), but they do not; (ii) in no sense has Palm proved that the hexagonal cell is the preferred (stable) mode of disturbance as $t \to \infty$. What he did show is that the hexagonal cell is one possible equilibrium state.

The former of these criticisms was discussed in a lecture at Stresa (Stuart 1960*b*) and it was concluded that Palm's published work contains errors which may invalidate the assertions he makes. (It was mentioned, and should be re-emphasized, that the principles of the analysis used by Palm are of considerable interest and importance.) Recently, in private correspondence, Prof. Palm has kindly notified us of some corrections to the algebra of his paper, and it appears that agreement is now established with the calculations of Malkus & Veronis (1958) and Gorkov (1957) for the case of constant viscosity. Prof. Palm has advanced the view that these corrections do not change the conclusions of his paper, namely, that the hexagonal cell is the preferred mode and that the fluid at the centre of the cell flows in the direction of increasing viscosity.

The present writers believe that Prof. Palm has not proved his assertions and, moreover, that they are not strictly true. (We find, for example, that the hexagonal mode can be preferred only if the variation of kinematic viscosity with temperature is sufficiently great.) Our work does, however, give some mathematical support to Palm's physical concepts, and it is our object to show where mathematics does and does not give such support. The central part of our analysis is a consideration of the solutions to Palm's revised pair of ordinary differential equations. The results of this analysis are summarized and discussed at the end of the paper.

2. Palm's analysis

Palm's work was an extension in two ways of the theory of finite-amplitude thermal convection: (i) more than one initial disturbance was considered to be present; and (ii) the viscosity was allowed to vary with temperature. The latter variation was incorporated because experiments of Graham (1933) and Tippelskirch (1956) showed it to be important, particularly with reference to the direction of circulation in the convection cells. We note, however, that the kinematic-viscosity variation with temperature was approximated, for mathematical simplicity, in a certain way (see (2.4) below). The validity of various other physical approximations made is discussed further in § 7, but for the present we can argue that the basic partial differential equations used by Palm form a better model of the non-linear problem than do equations (cf. Malkus & Veronis 1958; Gorkov 1957) which do not allow viscosity to vary with temperature.

Following earlier work, Palm assumed the initial temperature distribution to have a constant gradient between two horizontal surfaces. (For mathematical simplicity both of the latter were taken to be 'free', the boundary conditions being ones of zero vertical velocity, zero shear stress and specified temperature.) We feel that the assumption of an initially uniform temperature gradient can be justified on two grounds: (i) it leads to a meaningful and feasible problem which, in the linearized case, gives a critical Rayleigh number (for the onset of convection) in good agreement with experiment; (ii) since an equilibrium state of finite-amplitude convection involves a balance of diffusion and convection, it seems unlikely that this state will be affected by minor irregularities in the initial temperature distribution.

Palm showed that one effect of variation of viscosity is to lower the critical Rayleigh number. Proceeding to his non-linear analysis he considered an initial motion consisting of the sum of (i) a disturbance (vertical velocity) which, by rotation of axes, can be written as

$$A_{021} \cos 2ly \sin \lambda z \tag{2.1}$$

and (ii) of other disturbances ('noise') superimposed on this. (Here x and y are mutually perpendicular horizontal co-ordinates and z is the vertical co-ordinate chosen so that the boundary planes are at $z = 0, h$; also $\lambda = \pi/h$.) The amplitude A_{021} is a function of time t only. Palm pointed out that, to the second order, (2.1) can reinforce the disturbance

$$A_{111}(t) \cos kx \cos ly \sin \lambda z, \tag{2.2}$$

where

$$k^2 = 3l^2, \tag{2.3}$$

and be reinforced by it, provided A_{021} has the appropriate sign. (Condition (2.3) ensures that the 'overall' horizontal wave-number is the same for the two disturbances, which means that linear theory gives the same amplification rate for the two disturbances.) This effect occurs through the terms introduced by the variation of kinematic viscosity ν with temperature, namely through the variable terms of

$$\nu = \nu_0 + \gamma \cos [(\lambda/\beta)(\Theta + \theta)] = \nu_0 + \gamma \cos \lambda z + (\gamma\lambda/\beta)\, \theta \sin \lambda z, \tag{2.4}$$

where β denotes the initial uniform temperature gradient, $\Theta(z,t)$ the mean temperature, and $\theta(x,y,z,t)$ the departure of the temperature from this mean. Palm suggested that a mathematical analysis can be restricted to a study of the interaction and growth of (2.1) and (2.2) alone. His argument appears to have two stages: (i) disturbances are selected that have maximum amplification (which implies (2.3)) according to linearized theory; (ii) of that class of disturbances, (2.1) and (2.2) are chosen on the physical grounds that the second-order mechanism of mutual reinforcement will render them preferred over other (non-reinforced) amplitudes. Our view (see also § 7) is that a mathematical investigation, involving the interaction of many disturbances, is necessary before Palm's suggestion can be accepted with confidence. Nevertheless, a study of the simpler problem with two disturbances is a necessary preliminary to this further investigation. We therefore pursue, in a more complete way, the consequences of Palm's physical model.

After some complicated analysis (of the compressible Navier–Stokes, continuity and temperature equations, simplified by the Boussinesq approximation and subject to the physical approximations discussed in § 7), Palm obtained a pair of ordinary differential equations for the amplitudes $A_{111}(\equiv Y_1)$ and $A_{021}(\equiv Z_1)$ of (2.1) and (2.2). They are quoted here in revised form, following corrections communicated to us by Prof. Palm, and correspond to equations (6.17) and (6.18) of the original paper. Before giving these equations, however, it is desirable to mention the mathematical approximations involved in deriving them. Letting

$\Delta\beta$ denote the difference between the actual initial temperature gradient β and the critical initial gradient β_0 for the onset of convection, we assume that γ/ν_0 and $\Delta\beta/\beta_0$ are both very small. These conditions imply that the initial amplification rates of the disturbances are small. In such cases it is valid (as has been explained by Stuart 1960*a* in a related case) to neglect time differentials in the equations for the mean field and all components of the disturbance *except* in those for the fundamentals (2.1), (2.2), *provided* the object is to obtain the dominant part of the non-linear problem. Palm's work involves the neglect of such time differentials (see pp. 186, 190, 191) in the mean temperature and in the harmonics, but not in the fundamentals. On the other hand, we have derived equations (2.5) and (2.6) below by a formal expansion procedure analogous to that used by Watson (1960) and all the above approximations are natural concomitants of the formal scheme. (We note that, although time differentials of the mean temperature are neglected in computing coefficients of the terms kept in (2.5) and (2.6) below, the variation with time of the mean temperature field Θ is certainly taken into account, for Θ is given by a power series in Y_1 and Z_1.) To sum up, we agree with Palm that, in his (corrected) equations (2.5) and (2.6) below, only terms negligible for small γ/ν_0 and $\Delta\beta/\beta_0$, including higher powers of the amplitudes, are omitted.

The differential equations for the fundamental amplitudes are

$$\begin{aligned}\sigma Y_1'' + (\kappa+\nu_0)\,\sigma^2 Y_1' - 4g\alpha l^2(\Delta\beta)\,Y_1 \\ = -(\lambda\gamma/8)\,(48l^4 - 4\lambda^2 l^2 + 3\lambda^4)\,Y_1 Z_1 - R Y_1^3 - P Y_1 Z_1^2,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sigma Z_1'' + (\kappa+\nu_0)\,\sigma^2 Z_1' - 4g\alpha l^2(\Delta\beta)\,Z_1 \\ = -(\lambda\gamma/32)\,(48l^4 - 4\lambda^2 l^2 + 3\lambda^4)\,Y_1^2 - R_1 Z_1^3 - \tfrac{1}{2} P Y_1^2 Z_1,\end{aligned} \tag{2.6}$$

where corrections of order γ^2 in the coefficients of $(\Delta\beta)\,Y_1$, Y_1', $(\Delta\beta)\,Z_1$, Z_1', Y_1^2 and and $Y_1 Z_1$ are ignored, and the limits $\gamma \to 0$, $\Delta\beta \to 0$ have been taken in the coefficients of the third-order terms. Primes denote differentiation with respect to time t. The notation is as given in Palm's paper, except that R_1 in (2.6) is there given incorrectly by $4R$, $\tfrac{1}{2}P$ in (2.6) is incorrectly replaced by $\tfrac{1}{4}P$, and the coefficients of $Y_1 Z_1$ and Y_1^2 are given only for the special case $k^2 + l^2 = \tfrac{1}{2}\lambda^2$, which gives the overall wave-number appropriate to the critical Rayleigh number when terms of order γ^2 are neglected. Prof. Palm's revised calculations give

$$R = \frac{\lambda}{8}\left\{\kappa\sigma^2\left[P_1 + \frac{3}{4}\frac{G}{M}\right] + \frac{g\alpha\beta l^2}{\kappa}\left[\frac{P_1 + (\lambda/16\kappa\sigma)}{k^2+\lambda^2} + \frac{(3G/M) + (9\lambda/4\kappa\sigma)}{4(l^2+\lambda^2)} + \frac{2}{\kappa\lambda\sigma}\right]\right\}, \tag{2.7}$$

$$R_1 = \frac{1}{2}\frac{g\alpha\beta l^2}{\kappa^2\sigma}, \tag{2.8}$$

$$P = 4R - R_1, \tag{2.9}$$

where
$$P_1 = \frac{\lambda}{16}\frac{3\sigma\kappa(k^2+\lambda^2) + (g\alpha\beta k^2/\kappa\sigma)}{16\kappa\nu(k^2+\lambda^2)^3 - g\alpha\beta k^2}. \tag{2.10}$$

Other corrections noted by Prof. Palm include the following, where pages and equation numbers refer to his paper:

(i) in equations (6.7) add

$$\Theta_1 + \theta = \sum_{ijm} D_{ijm}(t)\cos ikx \cos jly \sin m\lambda z;$$

(ii) page 190, line 3, replace $\nu\nabla^2 w_z$ by $\nu\nabla^2 w_t$;

(iii) in (6.12) replace $-9\kappa l^2\lambda$ by $\frac{1}{8}\lambda\kappa\sigma^2$, and change sign of $\frac{1}{2}g\alpha\lambda l^2$;

(iv) in (6.14) the expression for D_{002} requires the addition of $-\beta A^2_{021}/8\lambda\kappa^2\sigma$.

It is convenient, before further study of equations (2.5), (2.6) is made, to convert to non-dimensional quantities by taking ν_0 as a reference viscosity, h as a length, κ/h as a velocity, $\nu_0\kappa/\alpha gh^3$ as a temperature and h^2/κ as a time. Then with

$$\alpha_1^2 = 4l^2h^2/\pi^2, \tag{2.11}$$

$$\gamma_1 = \gamma/\nu_0, \tag{2.12}$$

$$(Y, Z) = (h/\kappa)(Y_1, Z_1), \tag{2.13}$$

$$\mathscr{R} = \alpha\beta g h^4/\kappa\nu_0, \tag{2.14}$$

$$\mathscr{P} = \nu_0/\kappa, \tag{2.15}$$

equations (2.5) and (2.6) become

$$\pi^2(1+\alpha_1^2)Y'' + \pi^4(1+\mathscr{P})(1+\alpha_1^2)^2 Y' - \pi^2\alpha_1^2\mathscr{P}(\Delta\mathscr{R})Y = -\gamma_1 SYZ - R_0 Y^3 - P_0 YZ^2, \tag{2.16}$$

$$\pi^2(1+\alpha_1^2)Z'' + \pi^4(1+\mathscr{P})(1+\alpha_1^2)^2 Z' - \pi^2\alpha_1^2\mathscr{P}(\Delta\mathscr{R})Z = -\tfrac{1}{4}\gamma_1 SY^2 - R_{10}Z^3 - \tfrac{1}{2}P_0 Y^2 Z, \tag{2.17}$$

where

$$S = \tfrac{1}{8}\pi^5\mathscr{P}(3\alpha_1^4 - \alpha_1^2 + 3), \tag{2.18}$$

$$8R_0 = \pi^4\mathscr{P}^{-1}(1+\alpha_1^2)(P_{10}+P_{11}) + [\alpha_1^2\mathscr{R}/(3\alpha_1^2+4)(\alpha_1^2+4)] \times [(\alpha_1^2+4)P_{10} + (3\alpha_1^2+4)P_{11} + \tfrac{3}{4}\mathscr{P}(1+\alpha_1^2)^{-1}(2\alpha_1^4+13\alpha_1^2+14)], \tag{2.19}$$

$$8R_{10} = \mathscr{P}\mathscr{R}\alpha_1^2(1+\alpha_1^2)^{-1}, \tag{2.20}$$

$$P_0 = 4R_0 - R_{10}, \tag{2.21}$$

$$P_{10} = \tfrac{3}{16}[\pi^4(1+\alpha_1^2)(3\alpha_1^2+4) + \alpha_1^2(1+\alpha_1^2)^{-1}\mathscr{R}\mathscr{P}]/[\pi^4(3\alpha_1^2+4)^3 - 3\alpha_1^2\mathscr{R}], \tag{2.22}$$

$$P_{11} = \tfrac{9}{16}[\pi^4(1+\alpha_1^2)(4+\alpha_1^2) + \alpha_1^2(1+\alpha_1^2)^{-1}\mathscr{R}\mathscr{P}]/[\pi^4(4+\alpha_1^2)^3 - \alpha_1^2\mathscr{R}], \tag{2.23}$$

$$\Delta\mathscr{R} = \mathscr{R} - \mathscr{R}_c, \tag{2.24}$$

$\mathscr{R}_c$ denoting the critical Rayleigh number. It is consistent with our analysis so far to approximate the Rayleigh number in (2.19) to (2.23) by its critical value for $\gamma = 0$. It is then simple to show that R_0, P_0 and R_{10} are positive for all real α_1.

Equation (2.9) ensures that particular solutions of equations (2.5) and (2.6) may be obtained in the form

$$Y = \pm 2Z. \tag{2.25}$$

These correspond to hexagonal cells and, as Prof. Palm has pointed out privately, (2.9) ensures that solutions of equations (2.5) and (2.6) are of the same kind as the solutions of equations (6.17) and (6.18) of his paper. This remains true with our changes of the second-order coefficients in (6.17) and (6.18): in (2.5), the $Y_1 Z_1$ coefficient is still four times as large as the Y_1^2 coefficient in (2.6).

For the case $\gamma = 0$, there are special steady solutions of (2.5) and (2.6) corresponding to the so-called rectangular cells (with $k/l = \sqrt{3}$), two-dimensional cells, and hexagonal cells. The results for the two former cases are in agreement with the earlier results of Malkus & Veronis (1958, pp. 235–6 and 243–4), while the result for the hexagonal case is in agreement with the paper of Gorkov (1957, equation (14)), except for slight numerical discrepancies.

3. Further analysis

The next main problem is to study the stability of the steady (equilibrium) solutions of equations (2.16) and (2.17). To this end it is convenient first to rewrite the equations in the form

$$bY''+Y' = \epsilon_1 Y - a_1 YZ - RY^3 - PYZ^2, \tag{3.1}$$

$$bZ''+Z' = \epsilon_1 Z - \tfrac{1}{4}a_1 Y^2 - R_1 Z^3 - \tfrac{1}{2}PY^2 Z, \tag{3.2}$$

where the time has been re-defined to render the coefficients of Y' and Z' equal to unity and the zero suffixes of R_0, P_0, R_{10} have been dropped for convenience. We shall consider both ϵ_1 (which is proportional to the Rayleigh-number difference) and a_1 (which is proportional to the coefficient of the variation of viscosity with temperature) to be small parameters which can be varied independently.

We shall neglect the second-derivative terms as it can be shown that they do not affect the stability or instability of the steady solutions when ϵ_1 and a_1 are small. (If the equations are linearized about a given steady solution, the inclusion of the second derivatives merely adds two decaying exponential solutions to those solutions obtained by neglect of the second derivatives.) To accomplish this formally, we expand Y' and Z' in powers of Y and Z, up to terms of third order, and obtain

$$Y' = \epsilon Y - aYZ - RY^3 - PYZ^2, \tag{3.3}$$

$$Z' = \epsilon Z - \tfrac{1}{4}aY^2 - R_1 Z^3 - \tfrac{1}{2}PY^2 Z, \tag{3.4}$$

where the limits as $\epsilon_1, a_1 \to 0$ have been taken in the coefficients of the cubic terms. The coefficients ϵ and a differ slightly from ϵ_1 and a_1; in fact ϵ is given by the $O(\epsilon_1)$ root of

$$b\epsilon^2 + \epsilon - \epsilon_1 = 0, \tag{3.5}$$

and

$$a = a_1/(1+3k\epsilon). \tag{3.6}$$

The equilibrium solutions are

$$\text{I:} \quad Y = Z = 0, \tag{3.7}$$

$$\text{II}\,a,b\text{:} \quad Y = 0, \quad Z = \pm(\epsilon/R_1)^{\frac{1}{2}}, \tag{3.8}$$

$$\text{III}\,a,b\text{:} \quad Y = 2Z, \quad Z = (2T)^{-1}[-a \mp \sqrt{(a^2+4\epsilon T)}], \tag{3.9}$$

$$\text{IV}\,a,b\text{:} \quad Y = -2Z, \quad Z = (2T)^{-1}[-a \mp \sqrt{(a^2+4\epsilon T)}], \tag{3.10}$$

$$\text{V}\,a,b\text{:} \quad Z = -a/Q, \quad Y = \pm R^{-\frac{1}{2}}[\epsilon - R_1 a^2 Q^{-2}]^{\frac{1}{2}}, \tag{3.11}$$

where

$$T = P + 4R = 8R - R_1, \tag{3.12}$$

$$Q = P - R_1 = 2(2R - R_1). \tag{3.13}$$

Note that V is meaningless for

$$\epsilon Q^2 < R_1 a^2. \tag{3.14}$$

Solutions I to V can have physical meanings ascribed to them as follows. In accordance with experiment, and with a definition implied in Rayleigh's work (1916, p. 443), a cell is taken to be bounded in the (x, y)-plane by a contour through which no fluid flows and along which the vertical velocity has one sign. It then follows that I represents no motion, II represents cells of two-dimensional plan-form and III and IV represent cells of hexagonal plan-form. Case V represents closed cells except for $\epsilon = R_1 a^2 Q^{-2}$, when V is equivalent to II; when $\epsilon = a^2 Q^{-2}(4R+R_1)$, V is equivalent to certain of the hexagonal cases, and when $a \equiv 0$, V is the so-called rectangular cell. According to the above definition, however, the last case ($a = 0$, $Z = 0$; $Y \cos kx \cos ly$) does not represent a cell bounded by a rectangle, since the vertical velocity changes sign along possible rectangular contours in the (x, y)-plane; it represents, in general, a cell bounded by a certain curved contour.

The core of the analysis leading to (2.16) and (2.17) and thence to (3.3) and (3.4), is an expansion in powers of $Y(t)$ and $Z(t)$. For the expansions to converge, Y and Z must be small, which can now be seen to require that $\epsilon^{\frac{1}{2}}$ and a be small compared to 1. We have therefore been consistent (i) in keeping $O(\gamma)$ terms but ignoring $O(\gamma^2)$ terms in the quadratic coefficients (factors of YZ and Y^2) of (2.16) and (2.17), (ii) in ignoring $O(\gamma)$ and $O(\epsilon)$ terms, compared to the terms kept, in the third-order coefficients of (2.16) and (2.17), (iii) in taking limits as $\epsilon_1, a_1 \to 0$ in the third-order coefficients of (3.3) and (3.4), (iv) in approximating the coefficients of the linear terms, as described after (2.6) and (v) in neglecting higher powers.

4. Stability of the equilibrium solutions

A complete solution of (3.3) and (3.4) is not known, so to determine which of the equilibrium solutions will prevail we resort to a linearized stability analysis. This will enable us to classify the equilibrium solutions as nodes (stable or unstable), saddle points, etc. For this purpose we shall refer to the book by Stoker (1950, pp. 38–45).

Suppose that any equilibrium solution is represented by $Y = y_0$, $Z = z_0$. We then write

$$Y = y_0 + y, \quad Z = z_0 + z, \tag{4.1}$$

substitute in (3.3), (3.4), linearize, and obtain

$$y' = (\epsilon - az_0 - 3Ry_0^2 - Pz_0^2)\, y + (-ay_0 - 2Py_0 z_0)\, z, \tag{4.2}$$

$$z' = (-\tfrac{1}{2}ay_0 - Py_0 z_0)\, y + (\epsilon - 3R_1 z_0^2 - \tfrac{1}{2}Py_0^2)\, z, \tag{4.3}$$

which may be written in the form

$$dy/dz = (\alpha z + \beta y)/(\gamma z + \delta y). \tag{4.4}$$

Since $\alpha = 2\delta$, we have

$$(\beta - \gamma)^2 + 4\alpha\delta \geqslant 0 \tag{4.5}$$

and the solution is a saddle point or node according as (Stoker 1950)

$$\alpha\delta - \beta\gamma > 0 \text{ (saddle)}, \quad \text{or} \quad \alpha\delta - \beta\gamma < 0 \text{ (node)}. \tag{4.6 a, b}$$

In the latter case, the node is stable or unstable according as

$$\beta + \gamma < 0 \text{ (stable node)}, \quad \text{or} \quad \beta + \gamma > 0 \text{ (unstable node)}. \tag{4.7 a, b}$$

We do not consider the special cases where $\alpha\delta - \beta\gamma = 0$ and/or $\beta + \gamma = 0$. These are instances of 'structurally unstable' situations in which the qualitative picture is altered by an arbitrarily small change in the coefficients (Andronow & Chaikin 1949). The results here would have little relevance to our physical problem; in particular, except for structurally unstable cases, we can be sure that, for sufficiently small a and ϵ, our stability results will be unaffected by the approximations discussed at the end of § 3.

Determination of when the various cases in (4.6) and (4.7) occur is now straightforward, once it is observed that $2R - R_1$ and $8R - R_1$ can easily be shown positive for all real α_1 by using the substitution for $\mathscr{R}$ discussed under (2.24). The results are illustrated schematically in figure 1. Defining q by

$$q = a\epsilon^{-\frac{1}{2}}, \tag{4.8}$$

we see that the extreme right- and left-hand sides of the diagram represent the limits as q tends to $\pm\infty$, though a and ϵ themselves remain small compared with

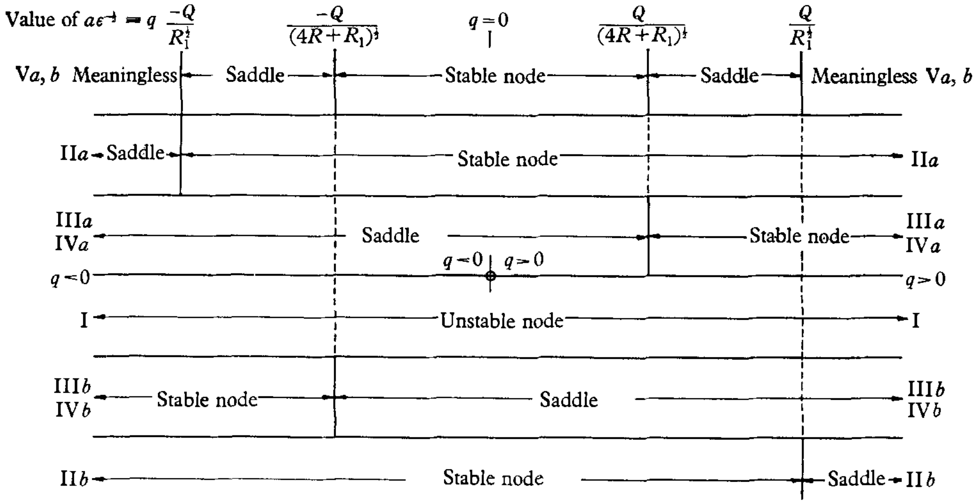

Figure 1. Schematic representation of stability of equilibrium solutions for $\epsilon > 0$. The diagram shows ranges of q for which the modes described below are stable (stable node) or unstable (unstable node or saddle point).
I: No motion; IIa, b: two-dimensional cells; IIIa, b and IVa, b: hexagonal cells; Va, b: closed cells except for $q = \pm QR_1^{-\frac{1}{2}}$, when they are two-dimensional. At $q = \pm Q(4R+R_1)^{-\frac{1}{2}}$ they are equivalent to certain of the hexagonal cells, while at $q = 0$ they take the form sometimes described as 'rectangular' ($Y \cos kx \cos ly$ with $Z \equiv 0$).

unity. The centre of the diagram, $q = 0$, represents the case when the viscosity does not vary with temperature. Here case V and the two-dimensional cells II are stable nodes, but the hexagonal cells III, IV are saddle points (and therefore unstable). Another point is that the two possible types of *closed* cell, hexagonal III, IV and case V, are never simultaneously stable nodes. On the other hand, a closed cell and a two-dimensional cell are stable nodes simultaneously for all values of q.

A major effect of the inclusion of the variation of viscosity with temperature is that, for sufficiently large values of q, the hexagonal cell changes character from saddle point to stable node; this is a most important feature brought out by our work. Given in figure 1 are the actual values of q which separate zones of different qualitative behaviour. These values are deduced from equations (4.2) and (4.3).

(Note, however, that at these values of q the situation is structurally unstable and the present discussion is not valid.)

Since, for a given q, there is always more than one stable node, Palm's contention that a hexagonal cell is the preferred mode is not necessarily true for large values of q. Moreover, when q is small, the hexagonal cell is a saddle (unstable) point so Palm's assertion cannot be true. By use of a little more mathematics together with some reasonable speculation, however, it is possible to elucidate the matter a little further. Some special solutions can be calculated, and these enable a plausible picture of the general solution to be obtained.

5. Behaviour of the solutions to (3.3), (3.4)

We consider $\epsilon > 0$. One solution of (3.3), (3.4) is

$$Y = 0, \quad Z = \pm\sqrt{[\epsilon C e^{2\epsilon t}/(1+R_1 C e^{2\epsilon t})]}, \tag{5.1}$$

where C is an arbitrary positive constant. A second solution is given by

$$Y = \pm 2Z, \quad Z' = \epsilon Z - aZ^2 - TZ^3, \tag{5.2}$$

whence, by separation of variables

$$|Z|\,|(Z-s)^r/(Z-r)^s|^{T'} = e^{\epsilon(t-t_0)}, \quad T' = T(a^2+4\epsilon T)^{-\frac{1}{2}}. \tag{5.3}$$

Here t_0 is an arbitrary constant, and r and s are the equilibrium values of Z with upper and lower signs, respectively, in (3.9) and (3.10). In both cases, choosing a different arbitrary constant corresponds to choosing a different time origin.

From (5.1) it can be seen that the solution tends to the equilibrium point IIa or IIb according as $Z \gtrless 0$. If we restrict consideration to $a > 0$, it can be shown from (5.3) that, for $Y = 2Z$ the solution tends to the equilibrium state IIIa or IIIb according as $Z \lessgtr 0$; similarly, for $Y = -2Z$, the solution tends to the equilibrium state IVa or IVb according as $Z \lessgtr 0$. (For $a < 0$ related results can be obtained.) In both (5.1) and (5.3) the solutions leave $Y = Z = 0$, as we would expect.

The integral curves given by (5.1), (5.2) and (5.3) are illustrated (for $a > 0$, $\epsilon > 0$, $1 \gg a^2 \gg \epsilon$) in figure 2 by the straight lines $Y = 0$, $Y = \pm 2Z$. The arrows illustrate the direction of development of the solutions as t increases. Also marked are the characters of the equilibrium points. At each equilibrium point, it is possible to determine the directions in which the integral curves may approach or leave (Stoker, 1950). The results, which should be read with reference to figure 2, are as follows:

UN I: all directions;

SN IIa: all solutions approach on $Y = 0$ except for one pair of singular solutions, which approach on $Z = (\epsilon/R_1)^{\frac{1}{2}}$;

S IIb: one pair of solutions approaches on $Y = 0$, and one pair of solutions leaves on $Z = -(\epsilon/R_1)^{\frac{1}{2}}$;

SN IIIa: all solutions approach on $Y = 2Z$ except for one pair of singular solutions, which approach with $dY/dZ = -1$;

S IIIb: one pair of solutions approaches on $Y = 2Z$, and one pair of solutions leaves with $dY/dZ = -1$;

SN IVa: all solutions approach on $Y = -2Z$, except for one pair of singular solutions, which approach with $dY/dZ = 1$;

S IVb: one pair of solutions approaches on $Y = 2Z$, and one pair of solutions leaves with $dY/dZ = 1$.

(In the above, UN denotes an unstable node, SN a stable node, and S a saddle point.)

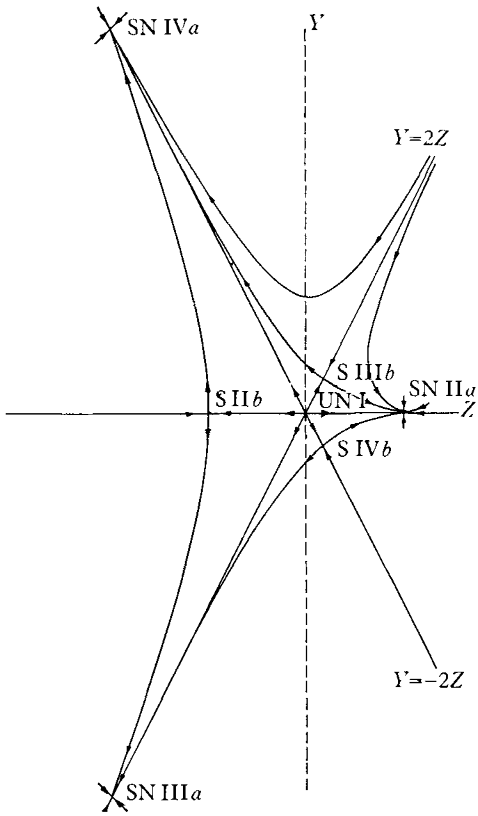

FIGURE 2. Trajectories for $a > 0$, $\epsilon > 0$, $1 \gg a^2 \gg \epsilon$: S, saddle point; SN, stable node; UN, unstable node. Straight lines are exact; curves are speculative.

With these properties it is possible to sketch curves of the kind shown in figure 2; the curves drawn are plausible but speculative. (One cannot be sure, for example, that the solution leaving S IIb for $Y > 0$ does not approach SN IVa along the singular path with $dY/dZ = 1$; it must approach SN IVa in some way, however, since no solutions to (3.3), (3.4) can tend to the line at infinity; similar reservations must be made with reference to the other curves in figure 2.) We are now in a position to suggest that the lines $Y = \pm 2Z\,(Z > 0)$ and $Y = 0\,(Z < 0)$ are lines of demarcation, in the following sense: (a) Above $Y = 0$ $(Z < 0)$ and $Y = 2Z$ $(Z > 0)$ trajectories tend to SN IVa; (b) below $Y = 0$ $(Z < 0)$ and $Y = -2Z$ $(Z > 0)$ trajectories tend to SN IIIa; (c) Between $Y = \pm 2Z$ $(Z > 0)$ trajectories tend to SN IIa. Thus we see that for a solution near the origin the stable node to which it tends depends upon the initial conditions under which the perturbation develops. Theoretically the two-dimensional mode may occur. This matter is discussed further in § 7.

If the hexagonal mode does occur, then the flow is downwards at the centre of the cell if $a > 0$ (because the stable nodes have $Z < 0$) and upwards at the centre of the cell if $a < 0$ (because the stable nodes have $Z > 0$). For example, the solution IIIa for a positive gives the vertical velocity (using non-dimensional quantities)

$$W = -[\{a+(a^2+4\epsilon T)^{\frac{1}{2}}\}/2T]\,[2\cos kx \cos ly + \cos 2ly]\sin \pi z, \tag{5.4}$$

which is negative at the cell centre $x = y = 0$. For the range of variables under consideration, $a > 0$ if and only if ν increases with temperature, and $a < 0$ if and only if ν decreases with temperature, so that our analysis suggests that the vertical flow is in the direction of increasing kinematic viscosity. This result was suggested by Palm from the physical requirement that the sign of Z be such that the second-order term destabilizes. We have shown that there are stable hexagonal equilibrium states with that sign of Z, thus supporting Palm's criterion, and we have determined the vertical-flow direction *at equilibrium*. Palm's argument does not appear to take third-order terms into account, but it is satisfying that we have obtained a similar result by so doing.

The above result concerning the direction of flow supports the explanation, which was first advanced by Graham (1933), of why the observed motion at the centre of the cell generally takes place in opposite directions in liquids and gases: the fluid flows in the direction of increasing viscosity, which occurs with increasing temperature in most gases but with decreasing temperature in most liquids. Graham's suggestion received support from Tippelskirch's (1956) important experiments with liquid sulphur near 153 °C, where its viscosity has a minimum; the vertical flow at the cell centre was in opposite directions according as the temperature lay above or below this value. Although the present theory suggests that kinematic viscosity, rather than viscosity, is the important parameter, this difference is probably unimportant in most experiments. Reservations are still necessary, however, because of the assumptions and approximations made.

6. Instability due to non-linear effects

An additional feature of equations (3.3) and (3.4) is that certain solutions can be amplified even for $\epsilon < 0$ (when the situation is stable according to linearized theory) if the amplitude of the disturbance is large enough. This provides an example of instability under subcritical conditions. (For a discussion of this possibility, see, for example, Stuart 1960a.) It can be seen from (3.8) and (3.11) that the equilibrium solutions II and V are meaningless (being purely imaginary) for $\epsilon < 0$. On the other hand, solutions III and IV have meaning if

$$a^2 + 4\epsilon T > 0. \tag{6.1}$$

The solution for the special cases $Y = \pm 2Z$ can be obtained from (5.3) in the form

$$|Z-r|^{sT'}/|Z|\,|Z-s|^{rT'} = \exp[|\epsilon|\,(t-t_0)], \tag{6.2}$$

where r, s and T' have the same definitions as in equation (5.3).

We can show that, for positive values of a in the range $a^2 > 4\,|\epsilon|\,T$, solutions IIIa and IVa are stable nodes while solutions IIIb and IVb are saddle points;

the reverse is true for a negative. The case when $a^2/|\epsilon|$ is large, and a is positive is illustrated in figure 3. The directions of the solutions on the straight lines $Y = \pm 2Z$, $Y = 0$ are given by (6.2) and (5.1), with C as a negative constant in the latter. The exact behaviour of the curves marked G is not known, but for our purposes it is sufficient to realize that solutions would tend to SN IVa if above the upper G curve and to SN IIIa if below the lower G curve. The nature of the solution (6.2) for $Y = 2Z$ is shown schematically as an inset in figure 3. The role of

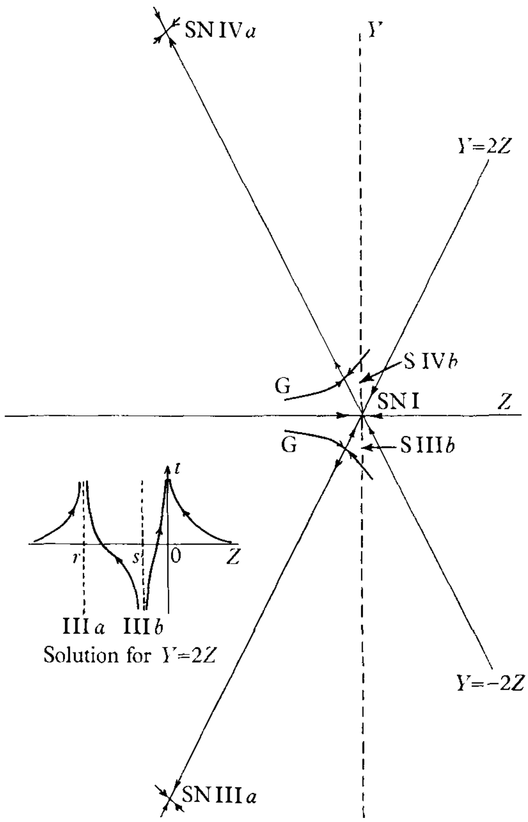

FIGURE 3. Trajectories for $a > 0$, $\epsilon < 0$, $1 \gg a^2 \gg |\epsilon|$: S, saddle point; SN, stable node; UN, unstable node. Straight lines are exact; curves G are approximate.

s is seen as a threshold amplitude above which disturbances amplify. It should be noted that the solution (6.2), like (5.3), is valid only for bounded Z, of order a^2 or ϵ; for larger amplitudes equations (3.3) and (3.4) are invalid because of higher-order terms omitted.

7. Summary and concluding remarks

We summarize our conclusions so far:

(1) Contrary to Palm's assertions, it is not possible to establish conclusively from his analysis, even as we have extended it, that 'when $t \to \infty$, the motion tends to a pattern consisting of hexagons'. For all values of the parameter a, representing the variation of viscosity with temperature, there is always another,

non-hexagonal, mode toward which solutions may tend as $t \to \infty$. The mode attained depends on the initial conditions for small amplitudes. Provided that a^2 is large enough, the final mode *may* be hexagonal, but for small values of a^2, including zero, it cannot be hexagonal. Other cells, including two-dimensional ones, also are stable in determined ranges of a^2. (Despite these mathematical conclusions, however, observation and physical reasoning suggest that the two-dimensional roll is unlikely to occur in practice; it is felt that the present result, that the two-dimensional roll can be a stable form, may have been introduced by the approximations and assumptions discussed below.) Even if the final mode *is* hexagonal, for earlier times the flow is a nondescript mixture of closed cells and rolls. This is in agreement with Benard's statement, quoted in Malkus & Veronis (1958, p. 256), that in his experiments (which may, however, have involved strong effects of surface tension) an initially 'disordered cellular régime' was quickly succeeded by a steady field of hexagons.

(2) The important role of the parameter a (or γ) is that, whereas for sufficiently small values of a^2 the hexagonal cells are saddle (unstable) points, for sufficiently large values of a^2 they are stable nodes.

(3) If the hexagonal mode occurs, the theory shows the vertical flow at the centre of a hexagonal cell to be in the direction of increasing kinematic viscosity. This agrees with experiment.

(4) Instability is theoretically possible for subcritical Rayleigh numbers ($\epsilon < 0$), provided that both γ (or a, cf. (6.1)) and the initial amplitude are large enough; here the final result as $t \to \infty$ is a hexagonal mode. It is not known whether this occurs in practice. For Rayleigh numbers below the critical, however, a certain type of instability has been observed, namely the 'columnar instabilities' of Chandra (1938). For a discussion giving a possible mechanism for this phenomenon see the Appendix to this paper.

Among the questions which remain to be resolved are the following: (i) Does use of the physically unrealizable 'free-free' boundary conditions invalidate the conclusions? (ii) Palm's work has shown the variation of viscosity with temperature to be very important. Are the conclusions invalidated by our omission of other small effects, such as surface tension, variation of thermal conductivity with temperature, viscous dissipation of energy, variation of density with temperature? Are the conclusions invalidated by the special form of the kinematic-viscosity relation with temperature? (iii) Can the choice of the original disturbances which are allowed to interact be rigorously justified? (iv) Can the equilibrium states obtained be proved stable against *all* possible disturbances?

We suspect that the answer to the first two questions is 'No' in the sense that the simplifications mentioned in (i) and (ii) are justified in a study showing *one* possible mechanism enabling the final establishment of hexagonal modes. That other mechanisms (and other modes) are possible is shown by the fact that unless a is large enough, hexagons are definitely unstable. When a is small, correct boundary conditions, for example, may well be essential. On the other hand, the agreement between the present theory and Graham's and Tippelskirch's observations (on the direction of flow at the cell centre) suggests that correct boundary conditions may not always be essential. It seems clear that Palm's analysis contains many of the important features necessary to describe Tippelskirch's results.

Point (iii) is both crucial and delicate and deserves careful discussion. The two main stages in Palm's argument are summarized in § 2. As far as stage (i) is concerned, it seems to us that one cannot argue that the disturbances will be selected so as to have the maximum amplification rate of linearized theory. For $\epsilon > 0$ ($\Delta\beta > 0$) there is a band of unstable wavelengths, and it is within the province of *non-linear* mechanics to determine which wavelength is preferred at finite amplitudes; a preliminary analysis of this kind is reported elsewhere (Segel 1962).

With reference to stage (ii), we think it is worth explaining further the concept of 'mutual reinforcement'. Consider a disturbance with two components, as follows:

$$\phi = A\cos kx\cos ly + B\cos mx\cos ny. \tag{7.1}$$

At second order, terms proportional to ϕ^2, $\phi_x^2+\phi_y^2$, $\phi_{xx}^2+2\phi_{xy}^2+\phi_{yy}^2$ appear, and give rise to factors like

$$\cos 2kx,\quad \cos 2ly,\quad \cos 2kx\cos 2ly,\quad \cos(k\pm m)x\cos(l\pm n)y,$$
$$\cos 2mx,\quad \cos 2ny,\quad \cos 2mx\cos 2ny. \tag{7.2}$$

To obtain reinforcement at second order the original modes of (7.1) must occur in (7.2). The condition $m = 0$, $n = 2l$, for example, ensures this. (But note that there are other, physically equivalent, possibilities of this kind.) If the wave-numbers (k^2+l^2) and (m^2+n^2) are the same, then $k^2 = 3l^2$; this is the case studied by Palm, and the consequences have been followed in this paper. (Another possibility is $k = 2m$, $l = 2n$, but this requires the Rayleigh number to be large enough for one unstable wave to have twice the wave-number of the other.) On the other hand, if one admits, as we think is necessary, the possibility of disturbances of different overall wave-numbers, then the problem cannot truly be limited to a study of the *two* disturbances (7.1).

A further difficulty inherent in Palm's suggestion is that reinforcement at second order also requires $aZ < 0$, as can be seen from (3.3), (3.4). If $aZ > 0$, Palm (1960) suggests that 'the sign of Z may be changed by displacing the frame of reference through half a wavelength along the y-axis', but this displacement also affects the expression for the Y disturbance. In fact if the Fourier spectrum of initial disturbances is specified (with say, $aZ > 0$), the sign of the component $Z\cos 2ly$ is given, and cannot be altered. (A possibility, however, is that, in regions where Z is of the proper sign for reinforcement, hexagons develop relatively quickly and then force this form on other regions by a kind of non-linear synchronization. It can be shown, in this connexion, that the hexagonal cell develops relatively more quickly than the two-dimensional cell.)

These arguments lead us to the conclusion that further work should, if possible, be done on the interaction of many disturbances of varying cell shapes and wave-numbers. As far as the present results are concerned, it seems possible that the stable character of the two-dimensional roll is more likely to be altered (by the inclusion of many other disturbances) than is the result that the hexagonal cells are stable if the viscosity variation is sufficiently great; but the mathematical validity of this suggestion remains to be assessed.

The difficulty in point (iv) is neatly illustrated on figure 2. Consider the hexagon equilibrium point IIIb. If we restrict our consideration to hexagonal disturbances with $Y = 2Z$, we see that this is a stable equilibrium point. But for other disturbances in the neighbourhood of this (saddle) point we have instability. One can say that what appears to be a stable node in one-dimensional 'disturbance space' is in fact a saddle point in two dimensions. We do not know whether IVa, which appears to be a stable node on the two-dimensional disturbance space we have considered, is a generalized saddle point in the infinite dimensional space of all possible disturbances. We would also like to point out that, *for the differential equations considered*, the apparent non-uniqueness which would be obtained in searching for equilibrium states by means of a time-independent steady analysis is resolved by consideration of the time-dependent problem with appropriate initial conditions.

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Appendix: Columnar instability

Columnar instability was first reported by Chandra (1938). It consists of a bursting upwards of buoyant fluid at irregularly spaced points, and has been observed also by Dassanayake (see Sutton 1950) and by de Graaf & van der Held (1953). (An apparently related phenomenon of downwards plunging sheets of fluid from a surface cooled by evaporation has been reported recently by Spangenberg & Rowland (1961); it seems likely, however, that surface tension was important in their experiments.) Columnar instability was found by Chandra to occur in air for Rayleigh numbers less than the critical (1708) for two rigid surfaces with constant viscosity, provided the distance h between the two planes was less than about 1 cm. In Dassanayake's experiments with carbon dioxide, columnar instability occurred at subcritical Rayleigh numbers for values of h less than about 7 mm. De Graaf & van der Held reported particular cases of columnar instability in air between two rigid surfaces at a Rayleigh number of 1400 and depths h of 5·5 and 6·9 mm.

It may be noted that Palm's work shows that the critical Rayleigh number of linear theory is lowered by the inclusion of variation of viscosity with temperature, and has the value

$$\mathscr{R}_c = (27\pi^4/4)\,(1-0{\cdot}1925\gamma^2/\nu_0^2). \tag{A.1}$$

This effect may account for part of the observed reduction of the critical Rayleigh number, but it cannot account for the form taken by columnar instability because Palm's convection cells are still regular. A further reduction of the theoretical critical Rayleigh number is afforded by the possibility (noted in this paper) of a subcritical instability for sufficiently large values of γ/ν_0; this effect, however, can at present only be evaluated from the non-linear *perturbation method* given in this paper, and on this basis the extra reduction of $\mathscr{R}_c$ can be shown to be much smaller than the reduction already given by Palm's linearized theory. Moreover, this subcritical effect also gives rise to a regular pattern of convection cells as the equilibrium state (cf. figure 3). We note, however, that the threshold-amplitude boundary (curves G of figure 3) might be passed locally in the fluid—perhaps even with a single cell as the basis of the convection pattern—since we can expect the finite disturbance to be stimulated by random variations in boundary conditions. This possibility might account for the random appearance of columnar instability. It is also conceivable that, for larger amplitudes of disturbance (outside the range of validity of this theory), subcritical instability might occur at much lower Rayleigh numbers and lead to less regular (perhaps columnar) patterns of equilibrium disturbance. These suggestions, it must be emphasized, are speculative and have no mathematical backing.

Since the Rayleigh number depends on the product of h and temperature difference ΔT $(= \beta_0 h)$, it is clear that the influence of viscosity variations will be greater in thinner layers because there both the temperature difference and the associated viscosity difference have to be greater in order to achieve the same Rayleigh number. Thus, as h is decreased, we may expect an increasing difference between the theoretical critical values of ΔT, calculated with and without the effect of variation of viscosity with temperature; a drop in the critical value of ΔT below the former of these values is, as we have stated, observed in experiment. The suggestion that 'the very great temperature gradients in (thinner) layers have some bearing on the lowering of the critical (Rayleigh) number' was first made by de Graaf & van der Held (1953).

We now explain the theoretical justification for the above remarks. The criterion (6.1) $a^2+4\epsilon T > 0$, $(\epsilon < 0)$ for subcritical instability with hexagonal cells can be reduced approximately to the form

$$Ay^2+\Omega xy-1 > 0, \tag{A. 2}$$

where
$$x = gh^3/\kappa\nu_0, \quad y = \Delta T/\overline{T}, \quad \Omega = \mathscr{R}_{c0}^{-1}. \tag{A. 3}$$

The symbol $\mathscr{R}_{c0}$ denotes the critical Rayleigh number in the absence of viscosity variations and A is a constant for a given gas. We restrict attention to gases, for which α^{-1} may be replaced by $\overline{T}$, the absolute temperature. Thus xy is the Rayleigh number. In deriving (A. 2) we have assumed that (i) the criterion (6.1) for subcritical instability is a representative one, even though the present model considers only a restricted class of disturbances, (ii) the wave-number is given by $k^2+l^2 = \pi^2/2h^2$, in Palm's notation, (iii) the viscous-variation coefficient γ/ν_0 may be approximated by $C\Delta T/2\overline{T}$, where C is a constant for a given gas, and (iv) the coefficients in (A. 2) may be evaluated according to the theory with two free boundaries. Experimental data suggests that, for air, $C = 1{\cdot}8$ and, for carbon dioxide, $C = 1{\cdot}95$, approximately. The coefficient A, which depends on C, also depends weakly on the Prandtl number, which is $0{\cdot}72$ for air and $0{\cdot}78$ for carbon dioxide. With these values it is found that, for air, $A = 0{\cdot}16$ and, for carbon dioxide, $A = 0{\cdot}19$, approximately.

If the coefficient A is set equal to zero, or if $\Omega x \to \infty$ for A fixed, equation (A. 2) yields the usual criterion,

$$xy > \Omega^{-1} \quad (\text{i.e.}\ \mathscr{R} > \mathscr{R}_{c0}), \tag{A. 4}$$

for instability in the absence of variation of viscosity with temperature. Two effects contribute to the coefficient A; the first is the reduction (according to Palm's linearized theory) of critical Rayleigh number due to variation of viscosity with temperature, and the second is the subcritical effect discussed in § 6 of this paper. The former of these two effects contributes about 95 % of the value of A, both for air and for carbon dioxide, and this is the basis of the remark made earlier that the reduction due to the subcritical effect is much smaller than that due to Palm's linear effect.

It can readily be shown from (A. 2) that, for A of order 0·2, the critical value of y suffers a 5 % or greater reduction (from the value given by linearized theory in the absence of viscosity variation) for Ωx of order 2 or less. On the other hand, Chandra's and Dassanayake's experiments show noticeable deviations (from the criterion given by linearized theory in the absence of viscosity variations) for Ωx of order 30 or less. In de Graaf & van der Held's experiments the corresponding value of Ωx was of order 3 or 5, but did not necessarily represent the largest value at which 'noticeable deviations' were possible. It appears, therefore, that quantitatively this theory and the experiments are not in agreement. It should be borne in mind, however, that (i) the theory is valid for 'free-free' boundary conditions, whereas the experiments had two rigid boundaries, and (ii) that the non-linear theory is valid only for small-amplitude perturbations. It is possible that these limitations, together with assumptions (i), (ii) and (iii) made earlier, have some bearing on the lack of quantitative agreement; but it is necessary to emphasize that this theory also does not predict the correct 'columnar' type of unstable behaviour.

From a qualitative point of view, equations (A. 2) and (A. 3) illustrate a behaviour very similar to that observed by Chandra and Dassanayake. Chandra plotted his experimental results in the form of $\eta = \Delta T/\kappa\nu_0 \overline{T}$ against $\xi = h$; but, for a given gas, with constant values of κ and ν_0, these variables are completely equivalent to the ones in (A. 3). Thus the behaviour given by (A. 2), namely a reduction in the critical value of y (from its value according to linearized theory without viscosity variation) as x decreases, is seen to be in qualitative agreement with the experimental results. It is also perhaps noteworthy that the closeness of the values of A in (A. 2) for air and carbon dioxide is qualitatively in accordance with the fact that the experimentally observed values of Ωx for 'noticeable deviations' are about the same for these two gases. It is qualitative features such as these which, we feel, justify our advancing the present theoretical argument.

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